Transport in two-dimensional scattering stochastic media: Simulations and models

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Classical transport of neutral particles in a purely scattering two-dimensional stochastic media is studied. Results of numerical Monte Carlo simulations of transport in two-dimensional stationary, binary, purely scattering stochastic media with Markovian mixing statistics are reported. Partial Markovian descriptions are proposed as models for the transport process inside the stochastic media. In these models, the composition of the media is correlated on a finite length scale. The results obtained from the models are in good agreement with the results obtained from the two-dimensional simulations.

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I. INTRODUCTION

A considerable amount of research dealt with classical transport of neutral particles and radiation in stochastic media [1–12]. The applications of this research are many, and include neutron transport in boiling water reactors, γ-ray and neutron flow through concrete shields, transport through molecular clouds and stellar atmospheres and radiative transfer in Rayleigh-Taylor unstable inertially confined fusion pellets.

In the current work we discuss time-independent monoenergetic transport, in a nonabsorbing stochastic media that does not contain internal particle sources. The transport equation for this process is written (based on the notation of neutron transport theory) [1]

$$\vec{\Omega}\cdot\vec{\nabla}\psi(\vec{r},\vec{\Omega})+\sigma_s\psi(\vec{r},\vec{\Omega})=\int\sigma_s(\Omega'\to\vec{\Omega})\psi(\vec{r},\vec{\Omega}')d\vec{\Omega}'. \tag{1}$$

$\psi(\vec{r},\vec{\Omega})$ is the angular flux, with $\vec{r}$ and $\vec{\Omega}$ denoting the spatial and angular variables respectively. σ_s and $\sigma_s(\vec{\Omega}'\to\vec{\Omega})$ are the total and differential macroscopic scattering cross sections, respectively.

The binary stochastic media examined is composed of grains of random size, shape, and placement, each filled with one of two materials. It is assumed that the cross sections of the constituent materials are known, but the information regarding the media's inner structure is known only in a statistical sense.

The heterogeneity of a media affects its transport properties. In a multidimensional scattering stochastic media, particles can bypass obstacles (opaque material grains) found along their path. This effect depends in general not only on the properties of the materials of the media, but also on its topology, which defines the possible particle paths. The effect of the heterogeneity thus differs between the various realizations of a stochastic medium. The study of transport in stochastic media deals with the average effect of heterogeneity over all possible realizations of the media, and with its fluctuations.

The transport equation is too complicated to be solved analytically, even when the inner structure of the medium is known. An estimate of the averaged transmission through a stochastic media can be obtained either using numerical simulations or using simplified models of the transport process.

Numerical results of transport simulations in stochastic media were reported for one-dimensional (1D) rod geometry [2,3], for layered planar geometry [3,4], and for media with cubic-shaped grains [5]. The results of the simulations showed that the transmission through stochastic multidimensional media is generally higher then the transmission through the equivalent homogenized (atomic-mixed) media [5].

Two simplified descriptions are widely used for the effective modeling of the transport process. The first is the 1D approach, where particles are restricted to move along a straight line, a feature that simplifies the mathematical complexity of the problem. When the materials in the media are both purely absorbing, or are both purely scattering, or they both have the same albedo, the transmission through each realization can be derived analytically, and the problem is reduced to that of averaging the nonlinear transmission function over the different realizations. Rigorous solutions for the 1D averaged transmission in stochastic media were found for purely absorbing [6–8] and purely scattering [9,10] media. These solutions were used to derive effective cross sections for the stochastic media. However, by restricting the transport to a straight path, the obstacle bypassing phenomenon is discarded. The 1D approach is thus limited to such problems where obstacle bypassing is not important. (This is the case, for example, in transport problems where the scattering is forward peaked, or when scattering is negligible. This is also the case in a layered planar geometry.)

In the second widely used model, the transport process in a given realization is taken to be a Markovian process, whose evolution depends only on its present state, and not on its past states [5,7,11]. Although transport in a heterogeneous medium is generally not a Markovian process, since the distances to material interfaces generally depend on the past trajectory, it has been suggested that the use of the Markovian assumption can produce a useful and simple model of

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particle transport in stochastic media. The Markovian assumption implies that the compositions of different segments along the particle's path are not correlated. This overemphasizes the obstacle bypassing effect, since obstacles ''disappear'' from the path at each collision. Thus the Markovian description is also limited to problems where obstacle bypassing is not important.

In the current article we present results of Monte Carlo transport simulations in purely scattering, two-dimensional, stochastic media with Markovian mixing statistics. Obstacle bypassing is an important feature in this problem, since there is a considerable amount of backscattering, and the transport range of the particles is not limited by an absorption mechanism.

A phenomenological model is proposed for the adequate modeling of obstacle bypassing in a 2D stochastic medium. The transport process in this model is partially Markovian, as the particle's trajectories are assumed to be partially correlated. A somewhat similar description for the transport process, containing partial path correlation, has been proposed earlier [13] as an efficient algorithm for transport in stochastic media. The results obtained from our partially Markovian model are found to be in good agreement with the results of the 2D simulations. The model results are also consistent with exact analytical results and approximate models obtained in the continuum limit for the effective properties of random media, including heat and electrical conductivity, magnetic permeability, and elasticity [16–21]. The model is thus an extension of the continuum results to problems where the geometrical length scale is comparable to the mean free path.

In Sec. II of this paper, the 2D media and the numerical simulations are described. In Sec. III, the partially Markovian model is formulated and tested.

II. SIMULATIONS OF TRANSPORT IN 2D STOCHASTIC MEDIA

The current work deals with 2D stochastic media composed of grains of random size, shape, and placement, each filled with one of two materials. Both materials are purely scattering, with cross sections denoted by σ_0 and σ_1. The averaged volume fractions of these materials are p_0 and p_1 $(=1-p_0)$, respectively. Due to the statistical nature of the stochastic media, the filling fractions deviate from the averaged fraction in a finite realization.

The mixing of the two materials is usually described using probability distribution functions for the material segment lengths—$f_i(l,\vec{\Omega},\vec{x})$—which define the probability that a segment of material i at space point $\vec{x}$ and in the direction $\vec{\Omega}$ will be of length l. In the current work we examine homogeneous, isotropic Markovian mixing statistics, for which $f_i(l,\vec{\Omega},\vec{x})$ is

$$f_i(l,\vec{\Omega},\vec{x})=f_i(l)=\frac{1}{\lambda_i}e^{-1/\lambda_i}, \tag{2}$$

λ_i being the average string length in material i. Markovian mixing statistics have the special property that the distance to the right (left) interface from any point inside the segment is independent of its distance from the left (right) interface.

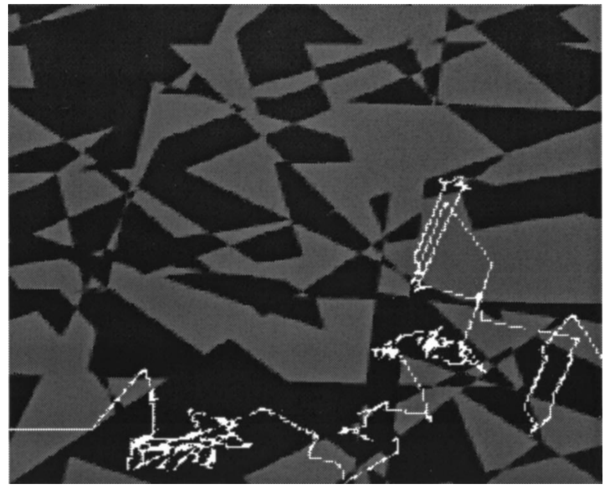

FIG. 1. A typical random realization of a 2D binary stochastic media with Markovian mixing statistics and a typical Monte Carlo path of a particle that passed through the medium.

The stochastic 2D realizations were constructed according to a procedure described by Switzer [14]: We sampled n lines with random distances from the center of the realization and with random directions. These lines divide the realization into convex polygonal cells. The string length distribution of these cells is Markovian and isotropic, with an average string length λ_C, that is inversely proportional to the number of lines. The material occupying each cell was sampled randomly according to the volume fractions p. Since adjacent cells can be filled with the same material, the averaged string length in a grain of material i is,

$$\lambda_i=\lambda_C+p_i\lambda_C+p_i^2\lambda_C+p_i^3\lambda_C+\cdots=\frac{\lambda_C}{1-p_i}. \tag{3}$$

A typical random realization constructed in this way is shown in Fig. 1.

The problem investigated in this paper is the longitudinal transmission through the medium, and its consequent characterization with an effective cross section. A Monte Carlo based transport code was used to generate ensembles of random rectangular realizations (for various stochastic media), whose depth and width were approximately $100\lambda_i$. A source of particles, emerging parallel to the $\hat{x}$ axis, was set at the left boundary of the media. Reflecting boundary conditions were applied at the transverse boundaries of the rectangle realizations. The calculation of each particle's path was done using Monte Carlo techniques, meaning that the distances between scattering interactions and the outcome directions were randomly sampled for each segment of the path. The particle history ended when the particle exited through the left or right faces of the medium. A typical particle path passing through a stochastic realization is also shown in Fig. 1.

The designated Monte Carlo code was used to calculate $T_{S,2D}(L)$—the average transmission through an ensemble of random realizations—by counting the relative number of particles reaching depth L from the left face of the media. The averaged transmission $T_{S,2D}(L)$ was then compared with the transmission through a 2D homogenized media, $T_{H,2D}(\sigma,L)$, and an effective 2D scattering cross section for the stochastic media was derived accordingly:

$$\sigma_{\mathrm{eff,2D}}(L)=\frac{T_{H,\mathrm{2D}}^{-1}(T_{S,\mathrm{2D}})}{L}. \tag{4}$$

Equation (4) implies that the transmission through a homogeneous medium with a scattering cross-section of $\sigma_{\mathrm{eff,2D}}(L)$ is equal to the average transmission through the random realizations of the stochastic medium.

Two sample problems are studied. In both problems the stochastic medium has equal material volume fractions ($p_0=p_1=0.5$) and the cross sections are $\sigma_0=20/11\,\mathrm{cm}^{-1}$ and $\sigma_1=2/11\,\mathrm{cm}^{-1}$ for the opaque and transparent material, respectively. In problem A, $\lambda_C=1$ cm, while in problem B $\lambda_C=30$ cm. (The units are actually arbitrary, since the problem is scale invariant as long as the dimensionless parameters $\lambda\sigma$ and L/λ are fixed.) The averaged (homogenized) cross section $\overline{\sigma}=p_0\sigma_0+p_1\sigma_1$ is equal to 1.0 for both problems. These problems were chosen since they correspond to different values of the important dimensionless parameter $\lambda_C\overline{\sigma}$. The dependence of the results on other dimensionless parameters, such as the contrast σ_0/σ_1 and the volume fractions (p_0,p_1), will be discussed later.

Figure 2 shows the normalized effective 2D cross sections $\sigma_{\mathrm{eff,2D}}(L)/\overline{\sigma}$ for the sample problems, as a function of the depth. Each problem's results are based on the calculation of more than 2.5 million particle histories inside a total of 50 000 2D realizations of the media. Effective cross sections obtained from models discussed in Sec. III are also plotted. In thin media, $\sigma_{\mathrm{eff,2D}}(L)$ approaches $\overline{\sigma}$. However, in thick media, the effective cross section is substantially lower (by more than 20% and 40% for the two sample problems, respectively), implying higher transmission.

Section II deals with models from which the results of Fig. 2 can be reproduced. It will be shown that the ability to bypass obstacles is the cause of the increase in the transmission through thick media.

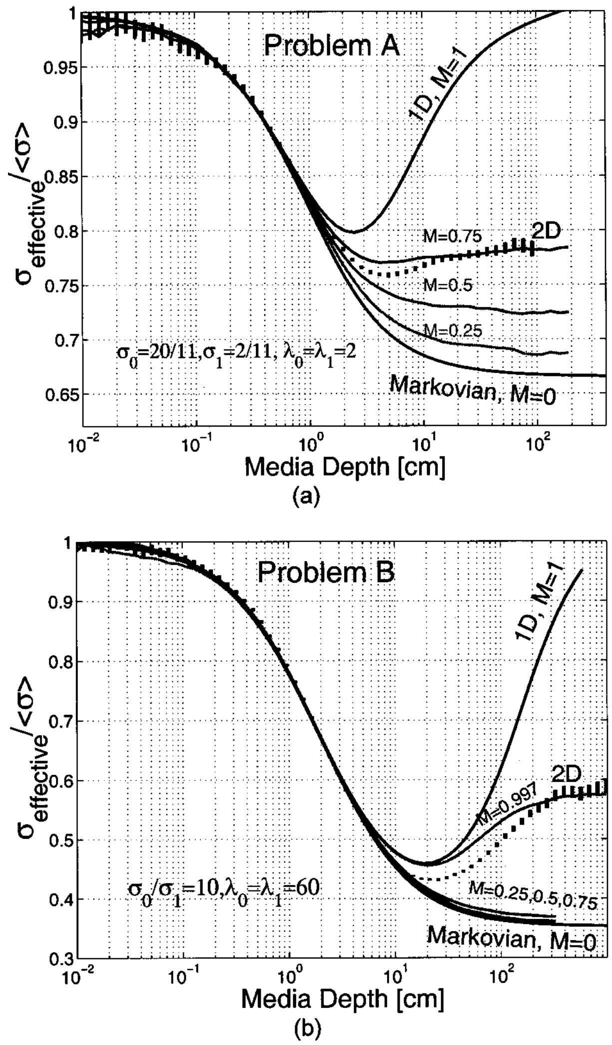

FIG. 2. The effective cross-section of the 2D stochastic medium, $\sigma_{\mathrm{eff,2D}}(L)$, vs the medium depth (with 2σ error bars), along with the results obtained from Monte Carlo simulations of partial Markovian transport processes with different values of the memory M. Among the partial Markovian results plotted are the results of the 1D approach and the results of the Markovian model.

III. ''PARTIAL MARKOVIAN'' MODELS

As mentioned earlier, the two most widely used simplifications of the transport process in stochastic media are the 1D and Markovian approaches. A significant difference between the two is that, in the 1D description, prescattering and post-scattering distributions of the material segments lengths along the particle path are fully correlated, while in the Markovian process there is no correlation between them.

By examining the path of a particle in a 2D scattering random medium (Fig. 1), it can be seen that as the particle's direction is changed by a scattering interaction; the material segment lengths along the new direction are only partially different from those along the previous one.

We are thus led to propose an approximation for the actual 2D process, based on a ''partial Markovian'' process. In such a process, prescattering and post-scattering segment length distributions are only partially correlated. There are many possible correlation forms. In the present work we examined a process in which, at each collision, either the composition of the media remains the same (total path correlation), or it changes randomly (zero path correlation).

The model was investigated using 1D Monte Carlo simulations, where the probability of the media's composition remaining the same after a scattering interaction was M (denoted hereafter as the ''memory''). In a no-memory interaction, occurring with a probability of $1-M$, all path information was lost, including the boundaries of the segment in which the interaction took place, and a new random realization was created around the particle. Such a process is illustrated in Fig. 3, where scattering interactions colored in black describe no-memory interactions. This process in an interpolation between the 1D approach ($M=1$) and a Markovian process restricted along a line ($M=0$).

Figure 2 presents effective cross sections of a partial Markovian scattering stochastic media, with different ''partial memories'' M (0, 0.25, 0.5, 0.75, and 1.0), as obtained for the sample problems. For problem B the line $M=0.997$ was added. The effective cross section increases with the memory (with the path correlation). The asymptotic (thick-media) effective cross sections of a partial Markovian process is lower than the averaged cross section, but higher than the Markovian one.

We note that a different Monte Carlo algorithm, with par-

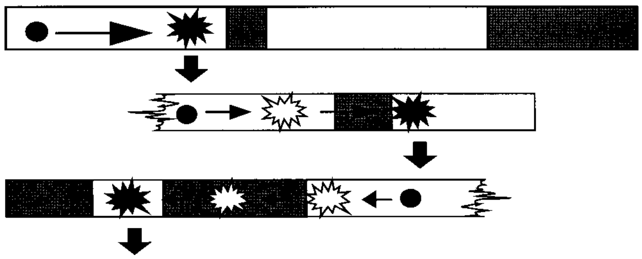

FIG. 3. A schematic description of a partial Markovian transport process in a 2D stochastic media. The transport begins in a 1D random realization. At each scattering interaction, either the particle continues to move in the same realizations (the interactions drawn in white), or all path memory is lost and the particle is transported into a different random realization (the interactions drawn in black). The relative probabilities for these two possibilities are governed by the memory M associated with the process.

tial path memory, was independently proposed as an efficient technique for approximating transport in stochastic media [16]. This algorithm retained memory of the distance between a particle's location and the boundaries of the surrounding material grain. Results obtained from this algorithm are similar to the ones shown in Fig. 2.

Next, we derive a method for the prediction of the value of M adequate for a specific 2D stochastic medium. The distance over which the path composition is correlated ("the correlation length") in the partial Markovian description depends on M. When $M=0$, it is simply the distance between consequent collisions, denoted by Δ. When M is close to unity, the correlation length can be estimated using simple diffusion (random walk) analysis: the path memory is lost after $N_C=1/(1-M)$ collisions on the average. The distance from the origin of this walk at which the loss of memory occurs is approximately

$$L_{\mathrm{cor}}(M)=\Delta\sqrt{N_c}=\Delta/\sqrt{1-M}. \tag{5}$$

Since Eq. (5) also reproduces the $M=0$ limit, we use it as a model for the correlation length for all M.

In a real 2D stochastic realization, the physical length scale over which the composition changes is λ_C. However, for very small grains, the path composition changes every collision ($L_{\mathrm{cor}}\approx\Delta$). We thus propose the following simple formula as a model for the correlation length:

$$L_{\mathrm{cor}}\approx\Delta+\lambda_C. \tag{6}$$

By equating the correlation lengths in Eqs. (5) and (6), and by associating Δ with the inverse of the effective cross section, $\Delta=1/\sigma_{\mathrm{eff}}$, for the partial memory M of a 2D media, we obtain

$$M=1-\frac{1}{(1+\sigma_{\mathrm{eff}}\lambda_C)^2}. \tag{7}$$

Since σ_{eff} is not known, the appropriate partial memory M in Eq. (7) is also unknown. Both can be derived iteratively by simulating several partial Markovian processes, each with an improved approximation for σ_{eff} and M (starting with the initial guess $\sigma_{\mathrm{eff}}=\overline{\sigma}$).

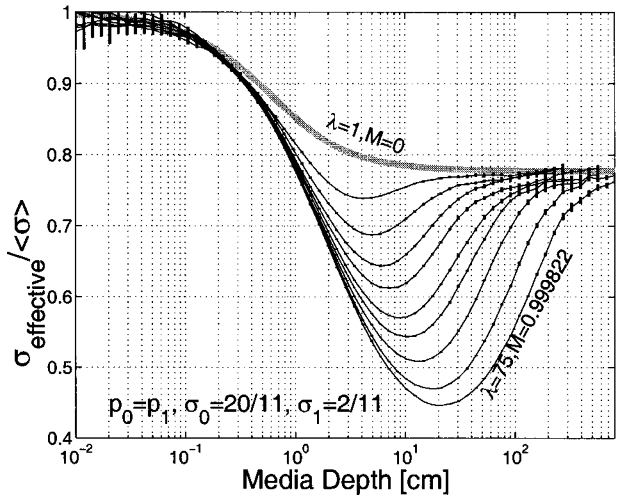

FIG. 4. Effective cross sections of equal c partial Markovian processes. The different lines correspond to problems with the average grain sizes $\overline{\lambda}=1$, 3, 5, 7.5, 10, 15, 20, 30, 50, and 75 and with the partial memories $M=0$, 8/9, 24/25, . . . , 0.9996, and 0.999822, respectively.

A better method for obtaining σ_{eff} and M is based on the fact, that the physical parameter governing the effect of the obstacle bypassing phenomena is the dimensionless ratio between the correlation length and the average grain length $c=L_{\mathrm{cor}}/\overline{\lambda}=1/(\overline{\lambda}\sigma_{\mathrm{eff}}\sqrt{1-M})$. If $c\gg1$, the results of the partial-Markovian process are similar to the results of the 1D process. If $c\ll1$, as is the case for $M=0$, 0.25, 0.5, and 0.75 in Fig. 2(b), the Markovian result is approached.

Partial Markovian problems having equal c (but different $\overline{\lambda}$ and M), have the same asymptotic effective cross-section. This statement was tested numerically in Fig. 4. We considered a set of problems in which $\overline{\lambda}$ takes values between 1.0 and 75, and M takes the corresponding values that result in $c=1/\sigma_{\mathrm{eff}}$ ($M=0$ for $\overline{\lambda}=1$, $M=0.999\,822$ for $\overline{\lambda}=75$, etc.). The effective cross sections of these partial Markovian problems are plotted in Fig. 4. All lines have approximately the same asymptotic effective cross section, in agreement with our statement. This asymptotic value can be derived analytically, from the Markovian problems $M=0$ and $\overline{\lambda}=1$ corresponding to the uppermost line in Fig. 4. Further analytical argumentation for the statement is given in the Appendix.

We can thus obtain the asymptotic effective cross section for a partial markovian process, using a renormalization step toward a Markovian description, keeping c (and the effective cross section) fixed:

$$\sigma_{\mathrm{eff}}(\lambda_C,M)=\sigma_{\mathrm{eff}}(\lambda_C\sqrt{1-M},0). \tag{8}$$

Using the asymptotic Markovian effective cross section [10]

$$\sigma_{\mathrm{eff}}(M=0)=\overline{\sigma}-\frac{v^2}{\hat{\sigma}} \tag{9}$$

where $\hat{\sigma}=p_0\sigma_1+p_1\sigma_0+1/\lambda_C$, and $v^2=p_0p_1(\sigma_0-\sigma_1)^2$, the following formula for the asymptotic effective cross-section of a partial Markovian description is obtained:

$$\sigma_{\mathrm{eff}}(M) \simeq \overline{\sigma} - \frac{v^2}{p_0\sigma_1 + p_1\sigma_0 + 1/\lambda_C\sqrt{1-M}}. \qquad (10)$$

After inserting Eq. (7) into Eq. (10), a quadratic equation for the effective cross-section is obtained, whose only physical solution ($\sigma_{\mathrm{eff}}>0$) is

$$\sigma_{\mathrm{eff}} = \tfrac{1}{2}[(\overline{\sigma}-\hat{\sigma}) + \sqrt{(\overline{\sigma}-\hat{\sigma})^2 + 4(\hat{\sigma}\overline{\sigma} - v^2)}]. \qquad (11)$$

This solution, in turn, can be inserted into Eq. (7) for the estimation of *M*. A somewhat similar spatial renormalization with a consequent change in correlation parameters was recently introduced in a study of diffusion processes in a fluctuating 1D lattice [15].

We now discuss some limits of the partial Markovian model. In the limit $\lambda\sigma\to 0$, the model approaches the Markovian description, which should be valid in this limit. Important observations can also be made in the continuum limit $\lambda\sigma\to\infty$, for a stochastic media whose depth *L* is much larger then the grain sizes. Equation (11) in this limit reproduces the symmetric-effective-medium-approximation [16–18], first proposed by Bruggeman [16], which is known to be in good agreement with experimental data. Our formula in this limit also satisfies the ''phase-interchange'' theorem [19] (which is an exact result), from which follows that when $p_0=p_1$, the effective cross section is $\sigma_{\mathrm{eff}}=\sqrt{\sigma_0\sigma_1}$. Finally, we note that if the extreme values $\Delta = 1/\sigma_0, 1/\sigma_1$ were used, the well-known Hashin-Shtrikman [20,21] upper and lower bounds for diffusion in stochastic (but homogeneous at large) media are obtained (in the 2D setting). We note that these bounds are analogs to a widely used composite-sphere-assemblage model (also known as the Maxwell-Garnett model) [17,20]. Thus, the current model is consistent with the known diffusion models, exact results, and bounds.

Figure 5 presents the asymptotic effective cross sections for a variety of problems. Plotted are the results of the 2D simulations, of the 1D and Markovian descriptions, and of the partial Markovian model. The parameters varied between the different problems are the average string length [Fig. 5(a)], the cross-sections ratio [Fig. 5(b)], and the volume fraction filled with the relatively transparent material [Fig. 5(c)]. These parameters span the possible stochastic media. The asymptotic results corresponding to the sample problems are enclosed in rectangles. For all the problems examined, the 2D effective cross section is lower than the 1D prediction for the asymptotic effective cross section ($\overline{\sigma}$), and higher than its Markovian counterpart ($\overline{\sigma}-v^2/\hat{\sigma}$). The use of the partial Markovian models reduces the discrepancy with the 2D results to several percent.

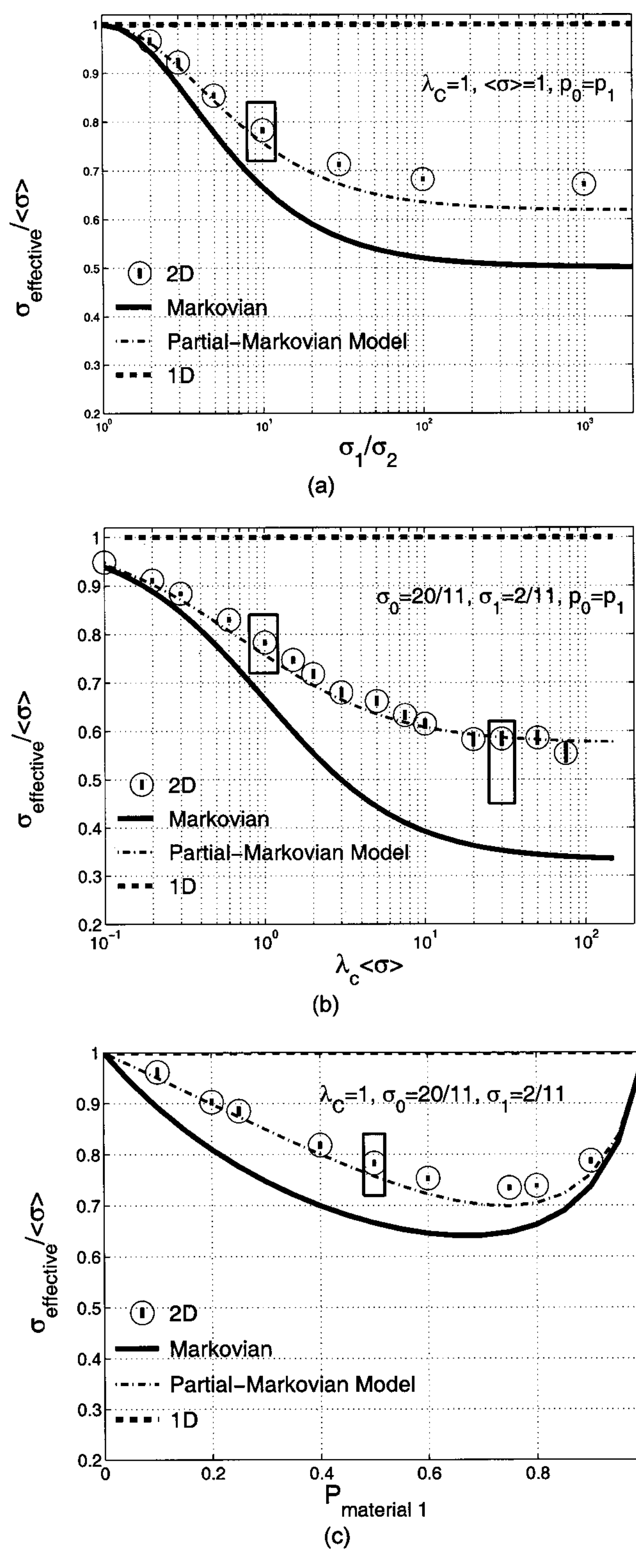

FIG. 5. Asymptotic effective cross sections as obtained from the 2D simulations (enclosed in a circle) and as obtained from the models, for a variety of stochastic media. Each subfigure shows results corresponding to different values of a dimensionless parameter: (a) the grain thickness $\lambda_C\overline{\sigma}$, (b) the contrast σ_0/σ_1, and (c) the volume fraction p_1.

IV. CONCLUSIONS

Transport in 2D, purely scattering stochastic media was discussed. Numerical Monte Carlo transport simulations in 2D purely scattering stochastic media were reported and used to quantify the multidimensional effect of obstacle bypassing for multiple problems.

A partial Markovian description of transport in stochastic media was proposed for the modeling of the transport process, reflecting the partial correlation between the paths before and after a 2D scattering interaction. The results of the partial Markovian descriptions were found to be in good agreement with the 2D simulation results, and were also found to reproduce known models, bounds, and exact results from the diffusion limit.

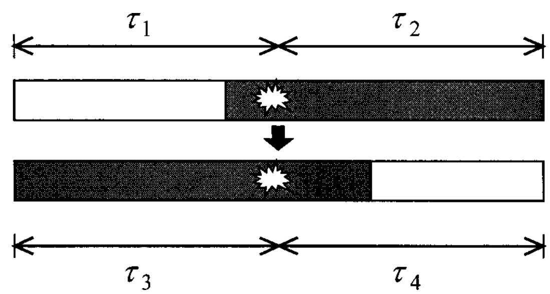

FIG. 6. A transition between two random realizations as a result of a no-memory interaction. A particle located at an internal point of a random realization, from which the optical distances to the medium's left and right faces are τ_1 and τ_2, respectively, passed to a different random realization due to a no-memory (Markovian) scattering interaction. In the new realization, the optical distances to the medium's left and right faces are different (τ_3 and τ_4).

APPENDIX

In this appendix we present argumentation for the claim that scaling the grain sizes in a partial Markovian transport process, while keeping c (the correlation length in terms of the average grain size) fixed, does not affect the asymptotic effective cross-section. We first consider the problem of transport along a 1D realization (the $c=\infty$ case). The effective cross section of the medium is simply a weighted average of the cross sections of the different segments, $\sigma_{\mathrm{eff}}=(\sigma_0\Sigma l_i+\sigma_1\Sigma l_j)/L$ (l_i refers to segments of material 0, l_j to segments of material 1, and L is the medium length). This effective cross section is scale invariant, since multiplying l_i, l_j, and L by a factor does not change the effective cross section.

We next consider the effect of a single no-memory scattering interaction: A particle lost its memory at some point in the interior of the media, whose optical depths from the left and right interfaces of the media are denoted by τ_1 and τ_2, respectively. In the new realization, the optical depths from the boundaries are denoted by τ_3 and τ_4, respectively. Such a transition is illustrated in Fig. 6.

The probability that, without the transition, the particle will reach the right interface of the medium (given that its direction after the scattering interaction is not specified) can be calculated as follows. The transmission $T(\tau)$ through a purely scattering media of depth τ, and the reflection $R(\tau)$ from this media, satisfy $T(\tau)=1-R(\tau)=2/(2+\tau)$. Suppose that the particle starts moving to the right after the scattering interaction. The probability that it will reach the right interface is

$$P_1=T(\tau_2)+[R(\tau_2)R(\tau_1)]T(\tau_2)+[R(\tau_2)R(\tau_1)]^2T(\tau_2)+\cdots=\frac{T(\tau_2)}{1-R(\tau_2)R(\tau_1)}. \tag{A1}$$

Taking into account the fact that a particle that was reflected from the right segment (τ_2) still has a chance of reaching the right interface. This chance depends on the probability of being reflected by the left segment (τ_1). Analogously, if the particle starts moving to the left, the probability of reaching the right interface of the medium is

$$P_2=R(\tau_1)T(\tau_2)+[R(\tau_1)R(\tau_2)]R(\tau_1)T(\tau_2)+[R(\tau_2)R(\tau_1)]^2R(\tau_1)T(\tau_2)+\cdots=\frac{R(\tau_1)T(\tau_2)}{1-R(\tau_2)R(\tau_1)}. \tag{A2}$$

Assuming isotropic scattering, the average probability of reaching the right interface is $P_{\mathrm{old}}=(P_1+P_2)/2=(1+\tau_1)/(2+\tau_1+\tau_2)$. Accordingly, if the particle passes to a new realization (with τ_3 and τ_4), the probability changes into $P_{\mathrm{new}}=(1+\tau_3)/(2+\tau_3+\tau_4)$. Thus the transition changed the probability of reaching the right interface by the factor

$$\frac{P_{\mathrm{new}}}{P_{\mathrm{old}}}=\frac{1+\tau_3}{2\tau_3+\tau_4}\,\frac{2+\tau_1+\tau_2}{1+\tau_1}. \tag{A3}$$

This factor is scale invariant under the condition $\tau_1,\tau_2,\tau_3,\tau_4\gg 1$. We can thus conclude that transitions between random realizations, occurring far from the medium faces, affect the transmission through the media (and hence the effective cross-section that asymptotically satisfies $\sigma_{\mathrm{eff}}\approx 2/T$) by a scale invariant factor.

When scaling the grain sizes in a partial Markovian transport process, the effective cross section remains fixed if the number of transitions between random realizations and their relative locations are fixed too. Keeping c— the correlation length in terms of the grain sizes—fixed secures that on the average, these conditions are satisfied.

We note that the condition $\tau_1,\tau_2,\tau_3,\tau_4\gg 1$ is not satisfied by all interior points, even when $L\rightarrow\infty$. This is due to the fact that the particle source is located on the left face of the medium, and not inside the medium. As it turns out, however, for those particles that have crossed the medium, the deviation from scale independence—in Eq. (A3)–cancels between transitions occurring close to both interfaces of the media.

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